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IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **43** (2010) 193001 (26pp) doi:10.1088/1751-8113/43/19/193001

TOPICAL REVIEW

Finite tight frames and some applications

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Received 16 July 2009, in final form 22 November 2009
Published 20 April 2010
Online at stacks.iop.org/JPhysA/43/193001

Abstract
A finite-dimensional Hilbert space is usually described in terms of an orthonormal basis, but in certain approaches or applications a description in terms of a finite overcomplete system of vectors, called a finite tight frame, may offer some advantages. The use of a finite tight frame may lead to a simpler description of the symmetry transformations, to a simpler and more symmetric form of invariants or to the possibility of defining new mathematical objects with physical meaning, particularly in regard with the notion of a quantization of a finite set. We present some results concerning the use of integer coefficients and frame quantization, and several examples, and suggest some possible applications.

PACS numbers: 03.65.Aa, 61.44.Br, 61.48.Gh
Mathematics Subject Classification: 42C99, 41A58

1. Introduction

Although, at first glance, a system described by a finite-dimensional Hilbert space looks much simpler than one described by an infinite-dimensional space, there is much more knowledge about the latter than the former. The continuous systems of coherent states have many applications [1, 31, 41] but the corresponding discrete version, usually called a frame, seems to be less used in quantum physics. Hilbert space frames, introduced by Duffin and Schaeffer in their work on nonharmonic Fourier series [16] were later rediscovered by Daubechies, Grossmann and Meyer in the fundamental paper [13]. Finite frames [1, 4, 5, 10, 19, 24] are useful in finite-dimensional quantum mechanics [46], particularly in quantum information [3, 33, 34], and play a significant role in signal processing (they give stable signal representations and allow modeling for noisy environments) [14]. Our aim is to present some results concerning the finite frames and their applications in physics, particularly in the context of quantization of finite sets. Particularly also, we try to prove that some mathematical

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methods used in modeling crystalline or quasicrystalline structures are in fact based on certain finite frames.

Each finite frame in a Hilbert space $\mathcal{H}$ defines an embedding of $\mathcal{H}$ into a higher dimensional Hilbert space (called a superspace), and conversely, each embedding of $\mathcal{H}$ into a superspace allows us to define some finite frames. The embedding into a superspace offers the possibility of defining some new mathematical objects, useful in certain applications. The construction of coherent states proposed by Perelomov in the case of Lie groups [40] admits a version for finite groups, and leads to some useful finite frames. Certain representations in terms of finite frames can be regarded as Riemann sums corresponding to the integrals occurring in some representations in terms of continuous frames.

The description of a physical system in terms of a finite frame allows us to associate a linear operator with a *classical* observable. The procedure, not necessarily a path to a quantum approach, can be regarded as an extended version of the Klauder–Berezin–Toeplitz quantization [6, 29, 30, 32] and represents a change of point of view in considering the physical system [18–23, 35].

The review is organized as follows. In section 2 we review some basic elements concerning the notion of tight frame in the form suitable for the applications in crystal physics and finite frame quantization we present throughout the review. We explain how Parseval frames are easily constructed by projection from higher dimensional spaces, and show how a superspace emerges naturally from the existence of a frame in a given Hilbert space. By following the analogy with the systems of coherent states we introduce the notion of normalized Parseval frame, define its proximity to an orthonormal basis in terms of a natural parameter η and describe some stochastic aspects. A Perelomov-like construction of frames through group representations is described at the end of the section. By taking into consideration the embedding into superspace, we investigate in section 3 the set of the elements which can be represented as a linear combination with integer coefficients of the frame vectors, and present some applications. We show in which way some simple crystalline structures in the plane or in space are naturally described with the aid of frames. Section 4 is devoted to what we call frame quantization of discrete variable functions. Frame quantization replaces such functions by matrices, introducing in this way non-commutative algebras of matrices. We present an interesting result issued from the stochastic aspects mentioned in section 2. We also introduce another parameter, ζ, expressing the distance of the 'quantum' non-commutative world issued from the frame quantization to the classical commutative one. We then illustrate our results concerning the proximity of the 'quantum non-commutativity' to the original 'classical' commutativity when the number of elements of a frame is larger by one than the dimension of the vector space.

2. Finite tight frames

2.1. *Finite frames*

Let $\mathbb{K}$ be the field $\mathbb{R}$ or $\mathbb{C}$, and let $\mathcal{H}$ be an N-dimensional Hilbert space over $\mathbb{K}$ with $\{|j\rangle\}_{j=1}^{N}$ a fixed orthonormal basis. A system of vectors $\{|w_i\rangle\}_{i=1}^{M}$ is a *finite frame* for $\mathcal{H}$ if there are constants $0 < A \leqslant B < \infty$ such that

$$A\|v\|^2 \leqslant \sum_{i=1}^{M} |\langle w_i|v\rangle|^2 \leqslant B\|v\|^2 \qquad \text{for all} \quad |v\rangle \in \mathcal{H}. \tag{1}$$

The frame operator

$$S|v\rangle = \sum_{i=1}^{M} |w_i\rangle\langle w_i|v\rangle \tag{2}$$

satisfies the relation

$$\langle v|Av\rangle = A\|v\|^2 \leqslant \sum_{i=1}^{M} |\langle w_i|v\rangle|^2 = \langle v|Sv\rangle \leqslant B\|v\|^2 = \langle v|Bv\rangle,$$

that is,

$$A\mathbb{I}_{\mathcal{H}} \leqslant S \leqslant B\mathbb{I}_{\mathcal{H}}$$

where $\mathbb{I}_{\mathcal{H}}$ is the identity operator. If $A = B$, the frame is called an *A-tight frame* and

$$S = A\mathbb{I}_{\mathcal{H}}.$$

A frame $\{|w_i\rangle\}_{i=1}^M$ is called an *equal norm frame* if $\|w_1\| = \|w_2\| = \cdots = \|w_M\|$. A 1-tight frame is usually called a *Parseval frame* and in this case

$$\sum_{i=1}^{M} |w_i\rangle\langle w_i| = \mathbb{I}_{\mathcal{H}}. \tag{3}$$

If $\{|w_i\rangle\}_{i=1}^M$ is an A-tight frame then $\left\{\frac{1}{\sqrt{A}}|w_i\rangle\right\}_{i=1}^M$ is a Parseval frame.

2.2. *Finite normalized Parseval frames*

Finite frames play a fundamental role in a wide variety of areas, and generally, each application requires a specific class of frames. In the case of finite frame quantization, we regard a Parseval frame as a finite family of coherent states. In order to improve the correspondence between the two notions we consider Parseval frames which do not contain the null vector and express their vectors in terms of some unit vectors.

Let $\{|w_i\rangle\}_{i=1}^M$ be a Parseval frame. Denoting

$$\kappa_i = \langle w_i|w_i\rangle \qquad \text{and} \qquad |u_i\rangle = \frac{1}{\sqrt{\kappa_i}}|w_i\rangle$$

the resolution of identity (3) becomes

$$\sum_{i=1}^{M} \kappa_i|u_i\rangle\langle u_i| = \mathbb{I}_{\mathcal{H}}. \tag{4}$$

We have

$$\langle v|w\rangle = \sum_{i=1}^{M} \kappa_i\langle v|u_i\rangle\langle u_i|w\rangle, \qquad \|v\|^2 = \sum_{i=1}^{M} \kappa_i|\langle u_i|v\rangle|^2 \tag{5}$$

for any $|v\rangle, |w\rangle \in \mathcal{H}$ and the well-known [25, 26, 47] relation

$$N = \sum_{j=1}^{N}\langle j|j\rangle = \sum_{j=1}^{N}\sum_{i=1}^{M} \kappa_i|\langle u_i|j\rangle|^2 = \sum_{i=1}^{M}\kappa_i\sum_{j=1}^{N}|\langle u_i|j\rangle|^2 = \sum_{i=1}^{M}\kappa_i. \tag{6}$$

In this review, by *normalized Parseval frame* in $\mathcal{H}$ we mean any system of vectors $\{|u_i\rangle\}_{i=1}^M$ satisfying the following two conditions:

(1) the vectors $|u_i\rangle$ are unit vectors, that is,

$$\langle u_i|u_i\rangle = 1, \qquad \text{for any} \quad i \in \{1, 2, \ldots, M\};$$

(2) there are $\{\kappa_i\}_{i=1}^M$ positive constants such that

$$\sum_{i=1}^M \kappa_i |u_i\rangle\langle u_i| = \mathbb{I}_{\mathcal{H}}. \tag{7}$$

If $\{|u_i\rangle\}_{i=1}^M$ is a normalized Parseval frame with the constants $\{\kappa_i\}_{i=1}^M$ then $\{\sqrt{\kappa_i}|u_i\rangle\}_{i=1}^M$ is a Parseval frame, and conversely, if $\{|w_i\rangle\}_{i=1}^M$ is a Parseval frame then $\left\{\frac{1}{\|w_i\|}|w_i\rangle\right\}_{i=1}^M$ is a normalized Parseval frame with the constants $\{\|w_i\|^2\}_{i=1}^M$. In the case $\kappa_1 = \kappa_2 = \cdots = \kappa_M$, relations (7) and (5) become [25, 26, 47]

$$\frac{N}{M}\sum_{i=1}^M |u_i\rangle\langle u_i| = \mathbb{I}_{\mathcal{H}}, \tag{8}$$

respectively,

$$\langle v|w\rangle = \frac{N}{M}\sum_{i=1}^M \langle v|u_i\rangle\langle u_i|w\rangle, \qquad \|v\|^2 = \frac{N}{M}\sum_{i=1}^M |\langle u_i|v\rangle|^2. \tag{9}$$

The frame is called a *finite equal norm Parseval frame* [7, 8] or a *finite normalized tight frame* [5].

2.3. Normalized Parseval frames versus orthonormal basis and stochastic aspects

Let us view the N components of the vector $|u_i\rangle$ with respect to the orthonormal basis $\{|j\rangle\}_{j=1}^N$ as the respective conjugates of N functions $i \mapsto \phi_j(i)$:

$$|u_i\rangle = \sum_{j=1}^N \bar{\phi}_j(i)|j\rangle \tag{10}$$

('bar' means complex conjugate). By using this expansion in the resolution of unity (7) we find the following orthogonality relations:

$$(\phi_j, \phi_k)_\kappa = \delta_{jk}, \tag{11}$$

with respect to the scalar product defined on the M-dimensional vector space of real or complex valued functions $i \mapsto \phi(i)$ on the set $X = \{1, 2, \ldots, M\}$ by

$$(\phi, \phi')_\kappa \stackrel{\text{def}}{=} \sum_{i=1}^M \kappa_i \bar{\phi}(i)\phi'(i). \tag{12}$$

By introducing the $N \times M$ matrix L with matrix elements

$$L_{ji} = \sqrt{\kappa_i}\bar{\phi}_j(i) = \sqrt{\kappa_i}\langle j|u_i\rangle, \tag{13}$$

we easily derive from (11) the equation

$$LL^\dagger = \mathbb{I}_{\mathcal{H}}. \tag{14}$$

Let us now express the pair overlaps $\langle u_i|u_{i'}\rangle$ in terms of the functions ϕ_j:

$$\langle u_i|u_{i'}\rangle = \sum_{j=1}^N \phi_j(i)\bar{\phi}_j(i') = (\mathrm{K}^{-1/2}L^\dagger L\mathrm{K}^{-1/2})_{ii'}, \tag{15}$$

where $\mathrm{K} \stackrel{\text{def}}{=} \mathrm{diag}(\kappa_1, \kappa_2, \dots, \kappa_M)$. If $M = N$, then (14) implies $L^\dagger = L^{-1}$ and so $\langle u_i | u_{i'} \rangle = \delta_{ii'}/\kappa_i$. The latter orthogonality relations together with (15) implies that $\kappa_i = 1$ for all i since the vectors $|u_i\rangle$'s are all unit. As expected, any family of N vectors satisfying (7) is an orthonormal basis.

Let us introduce the real $M \times M$ matrix U with matrix elements

$$U_{ij} = |\langle u_i | u_j \rangle|^2. \tag{16}$$

These elements obey $U_{ii} = 1$ for $1 \leqslant i \leqslant M$ and $0 \leqslant U_{ij} = U_{ji} \leqslant 1$ for any pair (i, j), with $i \neq j$.

Now we suppose that there is no pair of orthogonal elements, i.e. $0 < U_{ij}$ if $i \neq j$, and no pair of proportional elements, i.e. $U_{ij} < 1$ if $i \neq j$, in the frame. Then from the Perron–Frobenius theorem for (strictly) positive matrices, the rayon spectral $r = r(U)$ is > 0 and is a dominant simple eigenvalue of U. There exists a unique vector, v_r, $\|v_r\| = 1$, which is strictly positive (all components are > 0) and $U v_r = r v_r$. All other eigenvalues α of U lie within the open disk of radius $r: |\alpha| < r$. Since $\mathrm{tr}\, U = M$, and that U has M eigenvalues, one should have $r > 1$. The value $r = 1$ represents precisely the limit case in which all eigenvalues are 1, i.e. $U = \mathbb{I}$ and the frame is just an orthonormal basis of $\mathbb{C}^M$. It is then natural to view the number

$$\eta \stackrel{\text{def}}{=} r - 1 \tag{17}$$

as a kind of 'distance' of the frame to the orthonormality. The question is to find the relation between the set $\{\kappa_1, \kappa_2, \dots, \kappa_M\}$ of weights defining the frame and the distance η. By projecting on each vector $|u_i\rangle$ from both sides the frame resolution of unity (7), we easily obtain the M equations

$$1 = \langle u_i | u_i \rangle = \sum_{j=1}^{M} \kappa_j |\langle u_i | u_j \rangle|^2, \qquad \text{i.e.} \quad U v_\kappa = v_\delta, \tag{18}$$

where ${}^t v_\kappa \stackrel{\text{def}}{=} (\kappa_1\, \kappa_2 \dots \kappa_M)$ and ${}^t v_\delta \stackrel{\text{def}}{=} (1\, 1\, \dots\, 1)$ is the first diagonal vector in $\mathbb{C}^M$. In the 'uniform' case for which $\kappa_i = N/M$ for all i, i.e. in the case of a finite equal norm Parseval frame, which means that $v_\kappa = (N/M) v_\delta$, then $r = M/N$ and $v_r = 1/\sqrt{M} v_\delta$. In this case, the distance to orthonormality is just

$$\eta = \frac{M - N}{N}, \tag{19}$$

a relation which clearly exemplifies what we can expect at the limit $N \to M$.

Another aspect of a frame is the (right) stochastic nature of the matrix $P \stackrel{\text{def}}{=} U K$, evident from (18). The row vector $\varpi \stackrel{\text{def}}{=} {}^t v_\kappa / N = (\kappa_1/N \kappa_2/N \dots \kappa_M/N)$ is a stationary probability vector

$$\varpi P = \varpi. \tag{20}$$

As is well known, this vector obeys the ergodic property

$$\lim_{k \to \infty} (P^k)_{ij} = \varpi_j = \frac{\kappa_j}{N}. \tag{21}$$

2.4. Parseval frames obtained by projection

Let $\mathcal{E}$ be a finite-dimensional Hilbert space over $\mathbb{K}$, and let $\{|\varepsilon_1\rangle, |\varepsilon_2\rangle, \dots, |\varepsilon_M\rangle\}$ be an orthonormal basis in $\mathcal{E}$. A large class of tight frames can be obtained by projection [10].

Theorem 1. *If* $\{|\phi_j\rangle\}_{j=1}^N$ *is an orthonormal system in* $\mathcal{E}$, *then* $\{|w_i\rangle\}_{i=1}^M$, *where*

$$|w_i\rangle = \sum_{j=1}^N |\phi_j\rangle\langle\phi_j|\varepsilon_i\rangle \tag{22}$$

is a Parseval frame in the subspace $\mathcal{H} = \mathrm{span}\{|\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_N\rangle\}$, *that is,*

$$\mathcal{H} = \sum_{j=1}^N \mathbb{K}|\phi_j\rangle = \left\{ \sum_{j=1}^N \alpha_j|\phi_j\rangle \,\middle|\, \alpha_1, \alpha_2, \ldots, \alpha_N \in \mathbb{K} \right\}.$$

Proof. We obtain

$$\sum_{i=1}^M |w_i\rangle\langle w_i| = \sum_{i=1}^M \left(\sum_{j=1}^N |\phi_j\rangle\langle\phi_j|\varepsilon_i\rangle\right)\left(\sum_{k=1}^N \langle\varepsilon_i|\phi_k\rangle\langle\phi_k|\right)$$
$$= \sum_{j,k=1}^N \left(\sum_{i=1}^M \langle\phi_j|\varepsilon_i\rangle\langle\varepsilon_i|\phi_k\rangle\right)|\phi_j\rangle\langle\phi_k| = \mathbb{I}_{\mathcal{H}}.$$

□

The operator $\pi = \sum_{j=1}^N |\phi_j\rangle\langle\phi_j|$ is the orthogonal projector corresponding to $\mathcal{H}$ and $|w_i\rangle = \pi|\varepsilon_i\rangle$. If two orthonormal systems $\{|\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_N\rangle\}$ and $\{|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_N\rangle\}$ span the same subspace $\mathcal{H}$, then they define the same frame in $\mathcal{H}$. This means that the frame depends on the subspace $\mathcal{H}$ we choose, and not on the particular orthonormal system we use.

2.5. Embedding into a superspace defined by a Parseval frame

Let $\mathcal{H}$ be a Hilbert space over $\mathbb{K}$, $\{|j\rangle\}_{j=1}^N$ an orthonormal basis in $\mathcal{H}$, and let $\{|e_i\rangle\}_{i=1}^M$ be the canonical basis of $\mathbb{K}^M$. The following result, proved independently by Naimark and Han/Larson [10, 27] shows that any finite Parseval frame can be obtained by projection.

Theorem 2.

(a) If $\{|w_i\rangle\}_{i=1}^M$ *is a Parseval frame in* $\mathcal{H}$, *then the system* $\{|\phi_j\rangle\}_{j=1}^N$, *where*

$$|\phi_j\rangle = \sum_{i=1}^M |e_i\rangle\langle w_i|j\rangle = (\langle w_1|j\rangle, \langle w_2|j\rangle, \ldots, \langle w_M|j\rangle) \tag{23}$$

is an orthonormal system in $\mathbb{K}^M$.

(b) The Hilbert space $\mathcal{H}$ *can be identified with the subspace*

$$\tilde{\mathcal{H}} = \mathrm{span}\{|\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_N\rangle\}$$

of the superspace $\mathbb{K}^M$ *by using the isometry* $\mathcal{H} \longrightarrow \tilde{\mathcal{H}} : |v\rangle \mapsto |\tilde{v}\rangle$, *where*

$$|\tilde{v}\rangle = \sum_{j=1}^N |\phi_j\rangle\langle j|v\rangle = \sum_{i=1}^M |e_i\rangle\langle w_i|v\rangle = (\langle w_1|v\rangle, \langle w_2|v\rangle, \ldots, \langle w_M|v\rangle). \tag{24}$$

(c) The frame $\{|\tilde{w}_i\rangle\}_{i=1}^M$ *corresponding to* $\{|w_i\rangle\}_{i=1}^M$ *is the orthogonal projection of the orthonormal basis* $\{|e_i\rangle\}_{i=1}^M$:

$$|\tilde{w}_i\rangle = \pi|e_i\rangle \qquad \text{for any} \quad i \in \{1, 2, \ldots, M\}. \tag{25}$$

Proof.

(a) From (5) we deduce that $\langle\phi_j|\phi_k\rangle = \sum_{i=1}^{M}\langle j|w_i\rangle\langle w_i|k\rangle = \langle j|k\rangle = \delta_{jk}$.
(b) We obtain $|\tilde{v}\rangle = \sum_{j=1}^{N}|\phi_j\rangle\langle j|v\rangle = \sum_{j=1}^{N}\sum_{i=1}^{M}|e_i\rangle\langle w_i|j\rangle\langle j|v\rangle = \sum_{i=1}^{M}|e_i\rangle\langle w_i|v\rangle$.
(c) We have $\pi|e_i\rangle = \sum_{j=1}^{N}|\phi_j\rangle\langle j|w_i\rangle = |\tilde{w}_i\rangle$. □

The subspace $\tilde{\mathcal{H}}$ and the isometry $\mathcal{H}\longrightarrow\tilde{\mathcal{H}}$ have been defined by using an orthonormal basis $\{|j\rangle\}_{j=1}^{N}$ but they do not depend on the basis we choose. The representation $|\tilde{v}\rangle$ of $|v\rangle$ can be regarded as a discrete counterpart to the usual Fock–Bargmann representation [1].

2.6. Finite tight frames defined by using groups

Some useful frames can be defined in a natural way by using group representations [27]. Let $\{g:\mathcal{H}\longrightarrow\mathcal{H}|g\in G\}$ be an orthogonal (resp. unitary) irreducible representation of a finite group $\mathcal{G}$ in the real (resp. complex) n-dimensional Hilbert space $\mathcal{H}$, and let $|w\rangle\in\mathcal{H}$ be a fixed vector. The elements $g\in\mathcal{G}$ with the property

$$g|w\rangle = \alpha|w\rangle \tag{26}$$

where α is a scalar depending on g, form the stationary group $\mathcal{G}_w$ of $|w\rangle$.

Theorem 3. *If $\{g_i\}_{i=1}^{M}$ is a system of representatives of the left cosets of $\mathcal{G}$ on $\mathcal{G}_w$, then*

$$|w_1\rangle = g_1|w\rangle,\quad |w_2\rangle = g_2|w\rangle,\quad \ldots,\quad |w_M\rangle = g_M|w\rangle \tag{27}$$

form an equal norm tight frame in $\mathcal{H}$, namely

$$\sum_{i=1}^{M}|w_i\rangle\langle w_i| = \frac{M}{N}\|w\|^2\mathbb{I}_{\mathcal{H}}. \tag{28}$$

Proof. The operator $\Lambda:\mathcal{H}\longrightarrow\mathcal{H}$, $\Lambda|v\rangle = \sum_{i=1}^{M}|w_i\rangle\langle w_i|v\rangle$ is self-adjoint

$$\langle v'|(\Lambda|v\rangle) = \sum_{i=1}^{M}\langle v'|w_i\rangle\langle w_i|v\rangle = (\langle v'|\Lambda)|v\rangle$$

and therefore, it has a real eigenvalue λ. Since the eigenspace $\{|v\rangle;\,\Lambda|v\rangle = \lambda|v\rangle\}$ corresponding to λ is $\mathcal{G}$-invariant

$$\Lambda(g|v\rangle) = \sum_{i=1}^{M}|w_i\rangle\langle w_i|(g|v\rangle) = \sum_{i=1}^{M}g|w_i\rangle\langle w_i|v\rangle = g(\Lambda|v\rangle),$$

and the representation is irreducible we must have $\Lambda|v\rangle = \lambda|v\rangle$ for any $|v\rangle\in\mathcal{H}$. By using an orthogonal basis $\{|1\rangle,|2\rangle,\ldots,|N\rangle\}$ of $\mathcal{H}$ we obtain

$$N\lambda = \sum_{j=1}^{N}\langle j|\Lambda|j\rangle = \sum_{j=1}^{N}\sum_{i=1}^{M}\langle j|w_i\rangle\langle w_i|j\rangle = \sum_{i=1}^{M}\sum_{j=1}^{N}|\langle j|w_i\rangle|^2 = M\|w\|^2.$$ □

One can easily remark that the whole orbit

$$G|w\rangle = \{g|w\rangle|g\in G\}$$

is a tight frame, and more than that, any finite union of orbits is also a tight frame.

The relation

$$g(\alpha_1,\alpha_2) = \left(\alpha_1\cos\frac{2\pi}{n} - \alpha_2\sin\frac{2\pi}{n},\ \alpha_1\sin\frac{2\pi}{n} + \alpha_2\cos\frac{2\pi}{n}\right) \tag{29}$$

defines a representation of the cyclic group $\mathcal{C}_n = \langle g | g^n = e \rangle$ as a group of rotations of the plane, and for example, the orbit

$$\mathcal{C}_3\left(\sqrt{\tfrac{2}{3}}, 0\right) = \left\{\left(\sqrt{\tfrac{2}{3}}, 0\right), \left(-\tfrac{1}{\sqrt{6}}, \tfrac{1}{\sqrt{2}}\right), \left(-\tfrac{1}{\sqrt{6}}, -\tfrac{1}{\sqrt{2}}\right)\right\} \tag{30}$$

is a Parseval frame in $\mathbb{R}^2$. The relations

$$g(\alpha_1, \alpha_2, \alpha_3) = (-\alpha_1, -\alpha_2, \alpha_3), \qquad h(\alpha_1, \alpha_2, \alpha_3) = (\alpha_2, \alpha_3, \alpha_1) \tag{31}$$

define a representation of the tetrahedral group $\mathcal{T} = \langle g, h | g^2 = h^3 = (gh)^3 = e \rangle$ as a group of rotations of the space, and for example,

$$T\left(-\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right) = \left\{\left(-\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right), \left(\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}\right), \left(\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}\right), \left(-\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}\right)\right\} \tag{32}$$

is a Parseval frame in $\mathbb{R}^3$.

3. Integer coefficients

Let $\mathcal{H} = \mathbb{R}^N$ and $\{|w_i\rangle\}_{i=1}^M$, where

$$\begin{aligned} |w_1\rangle &= (w_{11}, w_{12}, \ldots, w_{1N}) \\ |w_2\rangle &= (w_{21}, w_{22}, \ldots, w_{2N}) \\ &\vdots \\ |w_M\rangle &= (w_{M1}, w_{M2}, \ldots, w_{MN}) \end{aligned}$$

be a Parseval frame in $\mathbb{R}^N$, that is,

$$\sum_{i=1}^{M} |w_i\rangle\langle w_i|v\rangle = |v\rangle \qquad \text{for any} \quad |v\rangle \in \mathbb{R}^N.$$

In view of theorem 2 the vectors

$$\begin{aligned} |\phi_1\rangle &= (w_{11}, w_{21}, \ldots, w_{M1}) \\ |\phi_2\rangle &= (w_{12}, w_{22}, \ldots, w_{M2}) \\ &\vdots \\ |\phi_N\rangle &= (w_{1N}, w_{2N}, \ldots, w_{MN}) \end{aligned}$$

form an orthonormal system in $\mathcal{E} = \mathbb{R}^M$, and the injective mapping (analysis operator)

$$T : \mathbb{R}^N \longrightarrow \mathbb{R}^M : |v\rangle \mapsto |\tilde{v}\rangle = (\langle w_1|v\rangle, \langle w_2|v\rangle, \ldots, \langle w_M|v\rangle),$$

which can be written as

$$\mathbb{R}^N \longrightarrow \mathbb{R}^M, \qquad T(\alpha_1, \alpha_2, \ldots, \alpha_N) = \alpha_1|\phi_1\rangle + \alpha_2|\phi_2\rangle + \cdots + \alpha_N|\phi_N\rangle,$$

allows us to identify $\mathbb{R}^N$ with the subspace

$$\tilde{\mathcal{H}} = \{\alpha_1|\phi_1\rangle + \alpha_2|\phi_2\rangle + \cdots + \alpha_N|\phi_N\rangle | \alpha_1, \alpha_2, \ldots, \alpha_N \in \mathbb{R}\}$$

of the superspace $\mathbb{R}^M$. The one-to-one mapping $\mathbb{R}^N \longrightarrow \tilde{\mathcal{H}} : |v\rangle \mapsto |\tilde{v}\rangle$ is an isometry

$$\langle \tilde{v}|\tilde{v}'\rangle = \langle v|v'\rangle, \qquad \|\tilde{v}\| = \|v\|$$

and $\{|\tilde{w}_i\rangle\}_{i=1}^M$ is a Parseval frame in $\tilde{\mathcal{H}}$ corresponding to $\{|w_i\rangle\}_{i=1}^M$

$$\sum_{i=1}^{M} |\tilde{w}_i\rangle\langle \tilde{w}_i|\tilde{v}\rangle = \sum_{i=1}^{M} |\tilde{w}_i\rangle\langle w_i|v\rangle = \sum_{i=1}^{M} T|w_i\rangle\langle w_i|v\rangle = T|v\rangle = |\tilde{v}\rangle.$$

The frame $\{|\tilde{w}_i\rangle\}_{i=1}^M$ is the orthogonal projection on $\tilde{\mathcal{H}}$ of the canonical basis

$$|e_1\rangle = (1, 0, 0 \ldots, 0), \qquad |e_2\rangle = (0, 1, 0, \ldots, 0), \qquad \ldots, \qquad |e_M\rangle = (0, 0, \ldots, 0, 1),$$

namely, by denoting $\pi = \sum_{j=1}^N |\phi_j\rangle\langle\phi_j|$, we have

$$|\tilde{w}_1\rangle = \pi|e_1\rangle, \qquad |\tilde{w}_2\rangle = \pi|e_2\rangle, \qquad \ldots, \qquad |\tilde{w}_M\rangle = \pi|e_M\rangle.$$

The matrix of π in terms of the canonical basis $\{|e_i\rangle\}_{i=1}^M$ is

$$\pi = \begin{pmatrix} \langle w_1|w_1\rangle & \langle w_1|w_2\rangle & \ldots & \langle w_1|w_M\rangle \\ \langle w_2|w_1\rangle & \langle w_2|w_2\rangle & \ldots & \langle w_2|w_M\rangle \\ \ldots & \ldots & \ldots & \ldots \\ \langle w_M|w_1\rangle & \langle w_M|w_2\rangle & \ldots & \langle w_M|w_M\rangle \end{pmatrix}. \tag{33}$$

The linear operator

$$\pi^\perp : \mathbb{R}^M \longrightarrow \mathbb{R}^M, \qquad \pi^\perp x = x - \pi x$$

is the orthogonal projector corresponding to the orthogonal complement

$$\tilde{\mathcal{H}}^\perp = \left\{ x = (x_1, x_2, \ldots, x_M) \,\middle|\, \sum_{i=1}^M x_i|w_i\rangle = 0 \right\} \tag{34}$$

of $\tilde{\mathcal{H}}$ in $\mathbb{R}^M$, and the vectors

$$\left|\tilde{w}_1^\perp\right\rangle = \pi^\perp|e_1\rangle, \qquad \left|\tilde{w}_2^\perp\right\rangle = \pi^\perp|e_2\rangle, \qquad \ldots, \qquad \left|\tilde{w}_M^\perp\right\rangle = \pi\left|e_M^\perp\right\rangle = \pi^\perp|e_M\rangle$$

form a frame $\left\{\left|\tilde{w}_i^\perp\right\rangle\right\}_{i=1}^M$ in $\tilde{\mathcal{H}}^\perp$ such that

$$|\tilde{w}_i\rangle + \left|\tilde{w}_i^\perp\right\rangle = |e_i\rangle \qquad \text{for any} \quad i \in \{1, 2, \ldots, M\},$$

called the *complementary frame* [27]. Particularly, one can remark that the complementary frame corresponding to an equal norm frame is an equal norm frame.

Each vector $|v\rangle \in \mathbb{R}^N$ can be written as a linear combination of the frame vectors $|w_i\rangle$:

$$|v\rangle = \sum_{i=1}^M |w_i\rangle\langle w_i|v\rangle$$

in terms of the *frame coefficients* $\langle w_i|v\rangle$. If $M > N$, then the representation of a vector $|v\rangle \in \mathcal{H}$ as a linear combination of the frame vectors is not unique, and we have

$$|v\rangle = \sum_{i=1}^M x_i|w_i\rangle,$$

that is, the relation

$$\sum_{i=1}^M x_i|w_i\rangle = \sum_{i=1}^M |w_i\rangle\langle w_i|v\rangle$$

which can be written as

$$\sum_{i=1}^M (x_i - \langle w_i|v\rangle)|w_i\rangle = 0,$$

if and only if

$$(x_1 - \langle w_1|v\rangle, x_2 - \langle w_2|v\rangle, \ldots, x_M - \langle w_M|v\rangle) \in \tilde{\mathcal{H}}^\perp,$$

that is, if and only if

$$(x_1, x_2, \ldots, x_M) \in (\langle w_1|v\rangle, \langle w_2|v\rangle, \ldots, \langle w_M|v\rangle) + \tilde{\mathcal{H}}^\perp.$$

From the last relation it follows that

$$|v\rangle=\sum_{i=1}^{M}x_i|w_i\rangle \quad\Longleftrightarrow\quad \pi(x_1,x_2,\ldots,x_M)=(\langle w_1|v\rangle,\langle w_2|v\rangle,\ldots,\langle w_M|v\rangle)$$

and the inequality obtained by Duffin and Schaeffer [16]:

$$|v\rangle=\sum_{i=1}^{M}x_i|w_i\rangle \quad\Longrightarrow\quad \sum_{i=1}^{M}(x_i)^2\geqslant\sum_{i=1}^{M}(\langle w_i|v\rangle)^2. \tag{35}$$

Each vector $|v\rangle\in\mathbb{R}^N$ admits a natural representation in terms of frame coefficients $\langle w_i|v\rangle$, but other representations may offer additional facilities. In certain applications it is advantageous [9] to replace the frame coefficients by *quantized coefficients*, i.e. by integer multiples of a given $\delta>0$. In this section we shall present some applications concerning the elements of a Hilbert space which can be written as a linear combination with integer coefficients of the vectors of a fixed frame.

3.1. Orthogonal projection of $\mathbb{Z}^M$ on a subspace of $\mathbb{R}^M$

Let E be a vector subspace of $\mathbb{R}^M$ and let $B_r(a)=\{x\in E|\ \|x-a\|<r\}$ be the open ball of center a and radius r. A set $D\subset E$ is *dense* in E if the ball $B_r(a)$ contains at least a point of D for any $a\in E$ and any $r\in(0,\infty)$. The set D is *relatively dense* in E if there is $r\in(0,\infty)$ such that the ball $B_r(a)$ contains at least a point of D for any $a\in E$. The set D is *discrete* in E if for each $a\in D$ there is $r\in(0,\infty)$ such that $D\cap B_r(a)=\{a\}$. The set D is *uniformly discrete* in E if there is $r\in(0,\infty)$ such that the ball $B_r(a)$ contains at most one point of D for any $a\in E$. The set D is a *Delone set* in E if it is both relatively dense and uniformly discrete in E. The set D is a *lattice* in E if it is both an additive subgroup of E and a Delone set in E. In order to describe the orthogonal projection of $\mathbb{Z}^M$ on E we will use the following result.

Theorem 4 [15, 43]. *Let $\Phi:\mathbb{R}^M\longrightarrow\mathbb{R}^L$ be a surjective linear mapping, where $L<M$. Then there are subspaces V, V' of $\mathbb{R}^L$ such that*

(a) $\mathbb{R}^L=V\oplus V'$
(b) $\Phi(\mathbb{Z}^M)=\Phi(\mathbb{Z}^M)\cap V+\Phi(\mathbb{Z}^M)\cap V'$
(c) $\Phi(\mathbb{Z}^M)\cap V'$ *is a lattice in* V'
(d) $\Phi(\mathbb{Z}^M)\cap V$ *is a dense subgroup of* V.

The subspace V in this decomposition is uniquely determined.

Theorem 4 allows us to describe the subsets

$$\pi(\mathbb{Z}^M)=\sum_{i=1}^{M}\mathbb{Z}|\tilde{w}_i\rangle=\left\{\sum_{i=1}^{M}n_i|\tilde{w}_i\rangle\,\middle|\,n_1,n_2,\ldots,n_M\in\mathbb{Z}\right\}$$

of $\tilde{\mathcal{H}}$ and

$$\pi^\perp(\mathbb{Z}^M)=\sum_{i=1}^{M}\mathbb{Z}\left|\tilde{w}_i^\perp\right\rangle=\left\{\sum_{i=1}^{M}n_i\left|\tilde{w}_i^\perp\right\rangle\,\middle|\,n_1,n_2,\ldots,n_M\in\mathbb{Z}\right\}$$

of $\tilde{\mathcal{H}}^\perp$. There are subspaces V, V' of $\tilde{\mathcal{H}}$ and subspaces W, W' of $\tilde{\mathcal{H}}^\perp$ such that

$$\begin{array}{ll}\tilde{\mathcal{H}}=V\oplus V' & \pi(\mathbb{Z}^M)=\pi(\mathbb{Z}^M)\cap V+\pi(\mathbb{Z}^M)\cap V'\\ \tilde{\mathcal{H}}^\perp=W\oplus W' & \pi^\perp(\mathbb{Z}^M)=\pi^\perp(\mathbb{Z}^M)\cap W+\pi^\perp(\mathbb{Z}^M)\cap W'\end{array} \tag{36}$$

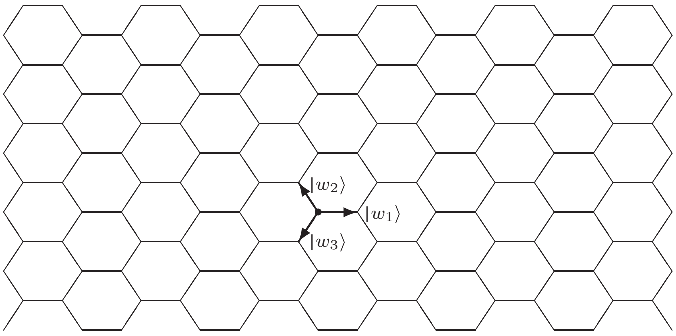

Figure 1. A fragment of the honeycomb lattice.

$\pi(\mathbb{Z}^M)\cap V'$ is a lattice in V', $\pi^\perp(\mathbb{Z}^M)\cap W'$ is a lattice in W', $\pi(\mathbb{Z}^M)\cap V$ is a dense subgroup of V and $\pi^\perp(\mathbb{Z}^M)\cap W$ is a dense subgroup of W.

We say that the starting frame $\{|w_i\rangle\}_{i=1}^M$ is a *periodic frame* if $V=\{0\}$, that is, if

$$\sum_{i=1}^M \mathbb{Z}|w_i\rangle = \left\{ \sum_{i=1}^M n_i|w_i\rangle \,\middle|\, n_1,n_2,\ldots,n_M\in\mathbb{Z}\right\}$$

is a lattice in $\mathcal{H}$. The frame $\{|w_i\rangle\}_{i=1}^M$ will be called a *quasiperiodic frame* if $W'=\{0\}$ and π restricted to $\mathbb{Z}^M$ is one-to-one. In this case, the collection of spaces and mappings

$$\begin{array}{ccccc} \tilde{\mathcal{H}} & \stackrel{\pi}{\longleftarrow} & \mathbb{R}^M & \stackrel{\pi^\perp}{\longrightarrow} & \tilde{\mathcal{H}}^\perp \\ & & \cup & & \\ & & \mathbb{Z}^M & & \end{array} \tag{37}$$

is a so-called *cut and project scheme* [39] and we can define the $*$-mapping as

$$\pi(\mathbb{Z}^M)\longrightarrow \tilde{\mathcal{H}}^\perp: \quad x\mapsto x^*=\pi^\perp((\pi|_{\mathbb{Z}^M})^{-1}x). \tag{38}$$

The projection π restricted to $\mathbb{Z}^M$ is one-to-one if and only if $\mathbb{Z}^M\cap\tilde{\mathcal{H}}^\perp=\{0\}$. The translations of $\tilde{\mathcal{H}}$ corresponding to the elements of $\mathbb{Z}^M\cap\tilde{\mathcal{H}}$ leave the set $\pi^\perp(\mathbb{Z}^M)$ invariant. If $\mathbb{Z}^M\cap\tilde{\mathcal{H}}$ contains a basis of $\tilde{\mathcal{H}}$, then the starting frame is a periodic frame.

3.2. Honeycomb lattice and diamond structure described in terms of frames

The symmetry properties of certain discrete sets can be simpler described by using a frame instead of a basis. Honeycomb lattice (figure 1) is a discrete subset $\mathcal{L}$ of the plane such that each point $P\in\mathcal{L}$ has three nearest neighbors forming an equilateral triangle centered at P. It can be described in a natural way by using the periodic Parseval frame (see (30))

$$|w_1\rangle=\left(\sqrt{\tfrac{2}{3}},0\right),\qquad |w_2\rangle=\left(-\tfrac{1}{\sqrt{6}},\tfrac{1}{\sqrt{2}}\right),\qquad |w_3\rangle=\left(-\tfrac{1}{\sqrt{6}},-\tfrac{1}{\sqrt{2}}\right)$$

as the set [12]

$$\mathcal{L}=\{n_1|w_1\rangle+n_2|w_2\rangle+n_3|w_3\rangle\,|\,(n_1,n_2,n_3)\in\mathbb{L}\},$$

where the subset

$$\mathbb{L}=\{n=(n_1,n_2,n_3)\in\mathbb{Z}^3\,|\,n_1+n_2+n_3\in\{0,1\}\}$$

of $\mathbb{Z}^3$ can be regarded as a mathematical model. The nearest neighbors of $n \in \mathbb{L}$ are

$$\begin{array}{l} n^1 = (n_1 + \nu(n), n_2, n_3) \\ n^2 = (n_1, n_2 + \nu(n), n_3) \\ n^3 = (n_1, n_2, n_3 + \nu(n)) \end{array} \qquad \text{where} \quad \nu(n) = (-1)^{n_1+n_2+n_3}.$$

The six points $n^{ij} = (n^i)^j$ corresponding to $i \neq j$ are the next-to-nearest neighbors, and one can remark that $n^{ii} = n$, $n^{ijl} = n^{lji}$, for any $i, j, l \in \{1, 2, 3\}$. The mapping

$$d : \mathbb{L} \times \mathbb{L} \longrightarrow \mathbb{Z} \qquad d(n, n') = |n_1 - n'_1| + |n_2 - n'_2| + |n_3 - n'_3|$$

is a distance on $\mathbb{L}$, and a point n' is a neighbor of order l of n if $d(n, n') = l$.

The symmetry group $\mathcal{G}$ of the honeycomb lattice is isomorphic with the group of all the isometries of the metric space $(\mathbb{L}, d)$, the group generated by the transformations

$$\begin{array}{ll} \mathbb{L} \longrightarrow \mathbb{L} : & (n_1, n_2, n_3) \mapsto (n_2, n_3, n_1) \\ \mathbb{L} \longrightarrow \mathbb{L} : & (n_1, n_2, n_3) \mapsto (n_1, n_3, n_2) \\ \mathbb{L} \longrightarrow \mathbb{L} : & (n_1, n_2, n_3) \mapsto (-n_1 + 1, -n_2, -n_3). \end{array}$$

Honeycomb lattice is a mathematical model for a graphene sheet and the use of the indicated frame leads to a simpler and more symmetric form for the $\mathcal{G}$-invariant mathematical objects occurring in the description of certain physical properties [12].

The diamond structure can be regarded as the three-dimensional analog of the honeycomb lattice. Each point P belonging to the diamond structure $\mathcal{D}$ has four nearest neighbors forming a regular tetrahedron centered at P. The diamond structure can be described in a natural way by using the periodic Parseval frame (see (32))

$$\begin{array}{ll} |w_1\rangle = \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), & |w_2\rangle = \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right), \\ |w_3\rangle = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right), & |w_4\rangle = \left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) \end{array}$$

of $\mathbb{R}^3$ as the set [11]

$$\mathcal{D} = \{n_1|w_1\rangle + n_2|w_2\rangle + n_3|w_3\rangle + n_4|w_4\rangle | (n_1, n_2, n_3, n_4) \in \mathbb{D}\},$$

where

$$\mathbb{D} = \{n = (n_1, n_2, n_3, n_4) \in \mathbb{Z}^4 | n_1 + n_2 + n_3 + n_4 \in \{0, 1\}\}.$$

The nearest neighbors of a point $n \in \mathbb{D}$ are

$$\begin{array}{l} n^1 = (n_1 + \nu(n), n_2, n_3, n_4) \\ n^2 = (n_1, n_2 + \nu(n), n_3, n_4) \\ n^3 = (n_1, n_2, n_3 + \nu(n), n_4) \\ n^4 = (n_1, n_2, n_3, n_4 + \nu(n)) \end{array} \qquad \text{where} \quad \nu(n) = (-1)^{n_1+n_2+n_3+n_4}.$$

The 12 points $n^{ij} = (n^i)^j$ corresponding to $i \neq j$ are the next-to-nearest neighbors, and one can remark that $n^{ii} = n$, $n^{ijl} = n^{lji}$, for any $i, j, l \in \{1, 2, 3, 4\}$. The mapping

$$d : \mathbb{D} \times \mathbb{D} \longrightarrow \mathbb{Z} \qquad d(n, n') = |n_1 - n'_1| + |n_2 - n'_2| + |n_3 - n'_3| + |n_4 - n'_4|$$

is a distance on $\mathbb{D}$, and a point n' is a neighbor of order l of n if $d(n, n') = l$.

The symmetry group O_h^7 of the diamond structure is isomorphic with the group of all the isometries of the metric space $(\mathbb{D}, d)$, the group generated by the transformations

$$\begin{array}{ll} \mathbb{D} \longrightarrow \mathbb{D} : & (n_1, n_2, n_3, n_4) \mapsto (n_3, n_4, n_2, n_1) \\ \mathbb{D} \longrightarrow \mathbb{D} : & (n_1, n_2, n_3, n_4) \mapsto (n_4, n_2, n_3, n_1) \\ \mathbb{D} \longrightarrow \mathbb{D} : & (n_1, n_2, n_3, n_4) \mapsto (-n_1 + 1, -n_2, -n_3, -n_4). \end{array}$$

Again the use of a frame leads to a simpler and more symmetric form for the O_h^7-invariant mathematical objects occurring in the description of certain physical properties [11].

3.3. An application to quasicrystals

The group $\mathcal{I}$ of all the rotations of $\mathbb{R}^3$ leaving a regular icosahedron centered at the origin invariant is called the *icosahedral group*. The 12 points

$$\pm(1,\tau,0),\quad \pm(-1,\tau,0),\quad \pm(-\tau,0,1),\quad \pm(0,-1,\tau),\quad \pm(\tau,0,1),\quad \pm(0,1,\tau)$$

where $\tau = (1+\sqrt{5})/2$, are the vertices of a regular icosahedron centered at origin. The rotations

$$r(\alpha,\beta,\gamma) = \left(\frac{\tau-1}{2}\alpha - \frac{\tau}{2}\beta + \frac{1}{2}\gamma,\ \frac{\tau}{2}\alpha + \frac{1}{2}\beta + \frac{\tau-1}{2}\gamma,\ -\frac{1}{2}\alpha + \frac{\tau-1}{2}\beta + \frac{\tau}{2}\gamma\right) \tag{39}$$
$$s(\alpha,\beta,\gamma) = (-\alpha,-\beta,\gamma)$$

satisfying the relation $r^5 = s^2 = (rs)^3 = \mathbb{I}_{\mathbb{R}^3}$ leave this regular icosahedron invariant, and therefore they define a representation of the icosahedral group in $\mathbb{R}^3$.

The stationary group $\mathcal{I}_w$ of $|w\rangle = \frac{1}{\sqrt{2(\tau+2)}}(1,\tau,0)$ is formed by the rotations $g \in \mathcal{I}$ with $g|w\rangle \in \{|w\rangle, -|w\rangle\}$, and we can choose the representatives $g_1, g_2, \ldots, g_6$ of the cosets of $\mathcal{I}$ on $\mathcal{I}_w$ such that

$$|w_1\rangle = g_1|w\rangle = \frac{1}{\sqrt{2(\tau+2)}}(1,\tau,0), \qquad |w_2\rangle = g_2|w\rangle = \frac{1}{\sqrt{2(\tau+2)}}(-1,\tau,0),$$
$$|w_3\rangle = g_3|w\rangle = \frac{1}{\sqrt{2(\tau+2)}}(-\tau,0,1), \qquad |w_4\rangle = g_4|w\rangle = \frac{1}{\sqrt{2(\tau+2)}}(0,-1,\tau),$$
$$|w_5\rangle = g_5|w\rangle = \frac{1}{\sqrt{2(\tau+2)}}(\tau,0,1), \qquad |w_6\rangle = g_6|w\rangle = \frac{1}{\sqrt{2(\tau+2)}}(0,1,\tau).$$

In view of theorem 3 the system $\{|w_i\rangle\}_{i=1}^6$ is a tight frame in $\mathbb{R}^3$. By direct computation one can prove that it is a quasiperiodic Parseval frame

$$\sum_{i=1}^{6} |w_i\rangle\langle w_i| = \mathbb{I}_{\mathbb{R}^3}.$$

It defines an embedding of $\mathcal{H} = \mathbb{R}^3$ in the superspace $\mathbb{R}^6$ and the set

$$\mathcal{Q} = \{x \in \pi(\mathbb{Z}^6) | x^* \in \pi^\perp([0,1]^6)\}, \tag{40}$$

defined by using the corresponding $*$-mapping is a quasiperiodic set.

The diffraction pattern corresponding to $\mathcal{Q}$ computed by using the Fourier transform is similar to the experimental diffraction patterns obtained in the case of certain icosahedral quasicrystals [17, 28]. Quasiperiodic sets corresponding to other quasicrystals can be obtained by starting from finite frames, and they help us to better understand the atomic structure of these materials.

3.4. Sequences of finite frames

Let $(f_n)_{n=0}^\infty$ be the Fibonacci sequence defined by recurrence as

$$f_0 = f_1 = 1, \qquad f_{n+1} = f_{n-1} + f_n$$

and let $\tau_n = f_{n+1}/f_n$. It is well known that $\lim_{n\to\infty} \tau_n = \tau$. The tetrahedral frame $\mathcal{T}(1,\tau,0)$ defined by using representation (31) coincides with the icosahedral frame $\mathcal{I}(1,\tau,0)$ defined by using representation (39)

$$\mathcal{T}(1,\tau,0) = \{(1,\tau,0), (-1,\tau,0), (-\tau,0,1), (0,-1,\tau), (\tau,0,1), (0,1,\tau), (-1,-\tau,0), (1,-\tau,0), (\tau,0,-1), (0,1,-\tau), (-\tau,0,-1), (0,-1,-\tau)\} = \mathcal{I}(1,\tau,0).$$

Therefore,

$$\lim_{n\to\infty} \mathcal{T}(1,\tau_n,0) = \lim_{n\to\infty}\{(1,\tau_n,0),(-1,\tau_n,0),(-\tau_n,0,1),(0,-1,\tau_n),(\tau_n,0,1),(0,1,\tau_n),\\ (-1,-\tau_n,0),(1,-\tau_n,0),(\tau_n,0,-1),(0,1,-\tau_n),(-\tau_n,0,-1),\\ (0,-1,-\tau_n)\} = \mathcal{I}(1,\tau,0),$$

that is, we can approximate the frame $\mathcal{I}(1,\tau,0)$ by using the periodic frames $\mathcal{T}(1,\tau_n,0)$.

The orbit $\mathcal{T}((1-t)(1,2,0)+t(1,\tau,0))$ of the tetrahedral group $\mathcal{T}$ is a frame in $\mathbb{R}^3$ for any $t\in[0,1]$. It can be regarded as a continuous deformation of the periodic frame $\mathcal{T}(1,2,0)$ into the icosahedral frame $\mathcal{I}(1,\tau,0)$.

The relation

$$\mathcal{R}_\theta(x,y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta) \tag{41}$$

defines an $\mathbb{R}$-irreducible two-dimensional representation of the multiplicative group

$$SO(2) = \left\{ \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} \middle|\, \theta\in[0,2\pi)\right\},$$

and the orbit $\{|\theta\rangle = (\cos\theta,\sin\theta)|\theta\in[0,2\pi)\}$ is a continuous frame

$$\frac{1}{\pi}\int_0^{2\pi} \mathrm{d}\theta|\theta\rangle\langle\theta| = \mathbb{I}_{\mathbb{R}^2}.$$

For any $n\in\mathbb{N}$ the orbit of $\mathcal{C}_n$ corresponding to $(1,0)$, namely,

$$\mathcal{C}_n(1,0) = \left\{ \left|\frac{2\pi}{n}k\right\rangle = \left(\cos\frac{2\pi}{n}k, \sin\frac{2\pi}{n}k\right) \middle|\, k\in\{0,1,\ldots,n-1\}\right\},$$

is a finite frame

$$\frac{2}{n}\sum_{k=0}^{n-1}\left|\frac{2\pi}{n}k\right\rangle\left\langle\frac{2\pi}{n}k\right| = \mathbb{I}_{\mathbb{R}^2},$$

and we have

$$\frac{2}{n}\sum_{k=0}^{n-1}\left|\frac{2\pi}{n}k\right\rangle\left\langle\frac{2\pi}{n}k\right| = \frac{1}{\pi}\frac{2\pi}{n}\sum_{k=0}^{n-1}\left|\frac{2\pi}{n}k\right\rangle\left\langle\frac{2\pi}{n}k\right| \overset{n\to\infty}{\longrightarrow} \frac{1}{\pi}\int_0^{2\pi}\mathrm{d}\theta|\theta\rangle\langle\theta|.$$

Therefore, we can regard the continuous frame $\{|\theta\rangle\}_{\theta\in[0,2\pi]}$ as the limit of the sequence of finite frames $(\mathcal{C}_n(1,0))_{n=3}^{\infty}$.

4. Frame quantization of discrete variable functions

4.1. Finite frame quantization

Let $\mathcal{X} = \{a_1,a_2,\ldots,a_M\}$ be a fixed finite set we regard as a set of data concerning a physical system. The space of all the functions $\varphi:\mathcal{X}\longrightarrow\mathbb{K}$ is a Hilbert space with the scalar product

$$\langle\varphi|\psi\rangle = \sum_{i=1}^{M}\overline{\varphi(a_i)}\psi(a_i) \tag{42}$$

(evidently, if $\mathbb{K}=\mathbb{R}$ then $\overline{\varphi(a_i)} = \varphi(a_i)$), and the isometry

$$l^2(\mathcal{X}) \longrightarrow \mathbb{K}^M: \quad \varphi\mapsto(\varphi(a_1),\varphi(a_2),\ldots,\varphi(a_M)) \tag{43}$$

allows us to identify the space $l^2(\mathcal{X})$ with the usual M-dimensional Hilbert space $\mathbb{K}^M$. The system of functions $\{\delta_1, \delta_2, \ldots, \delta_M\}$, where

$$\delta_i : \mathcal{X} \longrightarrow \mathbb{K}, \qquad \delta_i(a) = \begin{cases} 1 & \text{if} \quad a = a_i \\ 0 & \text{if} \quad a \neq a_i \end{cases}$$

is an orthonormal basis in $l^2(\mathcal{X})$

$$\varphi = \sum_{i=1}^{M} \langle \delta_i | \varphi \rangle \delta_i = \sum_{i=1}^{M} \varphi(a_i) \delta_i.$$

Let us select among the elements of $l^2(\mathcal{X})$ an orthonormal set $\{\phi_1, \phi_2, \ldots, \phi_N\}$ such that

$$\kappa_i = \sum_{j=1}^{N} |\phi_j(a_i)|^2 \neq 0, \qquad \text{for all} \quad i \in \{1, 2, \ldots, M\}$$

and let $\mathcal{H} = \text{span}\{\phi_1, \phi_2, \ldots, \phi_N\}$. In view of theorem 1, the elements

$$|u_i\rangle = \frac{1}{\sqrt{\kappa_i}} \sum_{j=1}^{N} \langle \phi_j | \delta_i \rangle \phi_j = \frac{1}{\sqrt{\kappa_i}} \sum_{j=1}^{N} \overline{\phi_j(a_i)} \phi_j, \qquad i \in \{1, 2, \ldots, M\} \tag{44}$$

form a normalized Parseval frame in $\mathcal{H}$, namely,

$$\sum_{i=1}^{M} \kappa_i |u_i\rangle\langle u_i| = \mathbb{I}_{\mathcal{H}}.$$

With each function $f : \mathcal{X} \longrightarrow \mathbb{R}$ which we regard as a *classical* observable we associate the linear operator

$$A_f : \quad \mathcal{H} \longrightarrow \mathcal{H}, \qquad A_f = \sum_{i=1}^{M} \kappa_i f(a_i) |u_i\rangle\langle u_i|. \tag{45}$$

This can be regarded as a Klauder–Berezin–Toeplitz-type quantization [6, 29, 30, 32] of f, the notion of quantization being considered here in a wide sense [18–23, 35]. The eigenvalues of the matrix A_f form the 'quantum spectrum' of f (by opposition to its 'classical spectrum' that is the set of its values $f(a_i)$). The function f is called *the upper (or contravariant) symbol* of A_f, and the function

$$\check{f} : \quad \mathcal{X} \longrightarrow \mathbb{R}, \qquad \check{f}(a_k) = \langle u_k | A_f | u_k \rangle = \sum_{i=1}^{M} \kappa_i f(a_i) |\langle u_i | u_k \rangle|^2 \tag{46}$$

is called *the lower (or covariant) symbol* of A_f. Since

$$\sum_{k=1}^{M} \kappa_i |\langle u_i | u_k \rangle|^2 = \|u_i\|^2 = 1,$$

the number $\check{f}(a_k)$ is a weighted mean of $f(a_1), f(a_2), \ldots, f(a_M)$, for any $k \in \{1, 2, \ldots, M\}$. In terms of the superspace, $\check{f}(a_k)$ can be regarded as a scalar product

$$\check{f}(a_k) = \langle (f(a_1), \ldots, f(a_M)), (\kappa_1 |\langle u_1 | u_k \rangle|^2, \ldots, \kappa_M |\langle u_M | u_k \rangle|^2) \rangle.$$

To a certain extent, a quantization scheme consists in adopting a certain point of view in dealing with $\mathcal{X}$. The presented frame quantization $f \mapsto A_f$ depends on the subspace $\mathcal{H} \subset l^2(\mathcal{X})$ we choose. The validity of the frame quantization corresponding to a certain subspace $\mathcal{H}$ is asserted by comparing spectral characteristics of A_f with data provided by specific protocol in the observation of the considered physical system. An interesting subject of a topological study is the triplet

$$[M \text{ values of } f] \leftrightarrow [N' \text{ eigenvalues of } A_f, N' \leqslant N] \leftrightarrow [M \text{ values of } \check{f}].$$

4.2. Probabilistic aspects of finite frame quantization

The relations

$$\sum_{j=1}^{N} |\langle \phi_j | u_i \rangle|^2 = 1 \qquad \text{for} \quad i \in \{1, 2, \ldots, M\}$$
$$\sum_{i=1}^{M} \kappa_i |\langle \phi_j | u_i \rangle|^2 = 1 \qquad \text{for} \quad j \in \{1, 2, \ldots, N\} \tag{47}$$

show that the considered normalized Parseval frame defines two families of probability distributions. This property can be interpreted in terms of a Bayesian duality [2].

If $\psi \in \mathcal{H}$ is such that $\|\psi\| = \sqrt{\langle \psi, \psi \rangle} = 1$, then

$$\sum_{i=1}^{M} |\sqrt{\kappa_i} \langle \psi | u_i \rangle|^2 = \sum_{i=1}^{M} \kappa_i |\langle \psi | u_i \rangle|^2 = \|\psi\|^2 = 1,$$

and hence, adopting the vocabulary of quantum measurement, $|\sqrt{\kappa_i}\langle \psi | u_i \rangle|^2$ can be viewed as the probability to find ψ in the state $|u_i\rangle$.

The trace of the operator A_f depends on the lower symbol

$$\begin{aligned}
\operatorname{tr} A_f = \sum_{k=1}^{N} \langle \phi_k | A_f | \phi_k \rangle &= \sum_{k=1}^{N} \sum_{i=1}^{M} \kappa_i \langle \phi_k | u_i \rangle \langle u_i | A_f | \phi_k \rangle \\
&= \sum_{i=1}^{M} \kappa_i \sum_{k=1}^{N} \langle u_i | A_f | \phi_k \rangle \langle \phi_k | u_i \rangle \\
&= \sum_{i=1}^{M} \kappa_i \langle u_i | A_f | u_i \rangle = \sum_{i=1}^{M} \kappa_i \check{f}(a_i).
\end{aligned}$$

An interesting problem in our finite frame quantization is to compare the starting function f with the lower symbol $\check{f}$. With the stochastic matrix notations of subsection 2.3, the relation

$$\check{f}(a_k) = \langle u_k | A_f | u_k \rangle = \sum_{i=1}^{M} \kappa_i f(a_i) |\langle u_i | u_k \rangle|^2$$

is rewritten as

$$\check{\mathbf{f}} = P\mathbf{f}, \tag{48}$$

with ${}^t\mathbf{f} \stackrel{\text{def}}{=} (f(a_1) f(a_2) \ldots f(a_M))$ and ${}^t\check{\mathbf{f}} \stackrel{\text{def}}{=} (\check{f}(a_1) \check{f}(a_2) \ldots \check{f}(a_M))$. This formula is interesting because it can be iterated:

$$\check{\mathbf{f}}^{[k]} = P^k \mathbf{f}, \qquad \check{\mathbf{f}}^{[k]} = P \check{\mathbf{f}}^{[k-1]}, \qquad \check{\mathbf{f}}^{[1]} \equiv \check{\mathbf{f}}, \tag{49}$$

and so we find from the property (21) of P that the ergodic limit (or 'long-term average') of the iteration stabilizes to the 'classical' average of the observable f defined as

$$\check{\mathbf{f}}^{[\infty]} = \langle f \rangle_{\mathrm{cl}} v_\delta, \qquad \langle f \rangle_{\mathrm{cl}} \stackrel{\text{def}}{=} \sum_{i=1}^{M} \frac{\kappa_i}{N} f(a_i). \tag{50}$$

4.3. The classical limit of finite frame quantization

We can evaluate the 'distance' between the lower symbol and its classical counterpart through the inequality

$$\|\check{\mathbf{f}} - \mathbf{f}\|_\infty \stackrel{\text{def}}{=} \max_{1\leqslant k\leqslant M} |f(a_k) - \check{f}(a_k)| \leqslant \|\mathbb{I} - P\|_\infty \|\mathbf{f}\|_\infty, \tag{51}$$

where the induced norm [37] on the matrix A is $\|A\|_\infty = \max_{1\leqslant i\leqslant M} \sum_{j=1}^{M} |a_{ij}|$. In the present case, because of the stochastic nature of P, we have

$$\|\mathbb{I} - P\|_\infty = 2\left(1 - \min_{1\leqslant i\leqslant M} \kappa_i\right). \tag{52}$$

In the uniform case, $\kappa_i = N/M$ for all i, we thus have an estimate of how far the two functions f and $\check{f}$ are: $\|\check{\mathbf{f}} - \mathbf{f}\|_\infty \leqslant 2(M-N)/M\|\mathbf{f}\|_\infty$. In the general case, we can view the parameter

$$\zeta \stackrel{\text{def}}{=} 1 - \min_{1\leqslant i\leqslant M} \kappa_i, \tag{53}$$

as a distance of the 'quantum world' to the classical one, of non-commutativity to commutativity, or again of the frame to orthonormal basis, such as the distance $\eta = r - 1$ introduced in subsection 2.3. Another way to check that $\zeta = 1 - N/M \to 0$ means, in the uniform case $\kappa_i = N/M$ for all i, that we go back to the classical spectrum of the observable f results from the following relations. We have

$$\frac{N}{M}\sum_{j\neq k} |\langle u_j|u_k\rangle|^2 = \|u_k\|^2 - \frac{N}{M}|\langle u_k|u_k\rangle|^2 = 1 - \frac{N}{M},$$

and from the relation

$$\check{f}(a_k) = \frac{N}{M}\sum_{j=1}^{M} f(a_j)|\langle u_j|u_k\rangle|^2 = \frac{N}{M} f(a_k) + \sum_{j\neq k} f(a_j)|\langle u_j|u_k\rangle|^2,$$

we obtain

$$\left(1 - \frac{N}{M}\right)\min_j f(a_j) \leqslant \check{f}(a_k) - \frac{N}{M} f(a_k) \leqslant \left(1 - \frac{N}{M}\right)\max_j f(a_j).$$

Finally, note the estimates

$$|f(a_k) - \check{f}(a_k)| = \left|\sum_{i=1}^{M} \kappa_i (f(a_k) - f(a_i))|\langle u_i|u_k\rangle|^2\right| \leqslant \max_i |f(a_k) - f(a_i)|,$$

whence

$$\|\check{\mathbf{f}} - \mathbf{f}\|_\infty \leqslant \max_{i,k} |f(a_k) - f(a_i)|. \tag{54}$$

4.4. Frames defined by using eigenvectors of non-commuting operators

Let $A, B : \mathbb{H} \longrightarrow \mathbb{H}$ be two operators on a Hilbert space $\mathbb{H}$, which are diagonalizable operators with orthogonal eigenvectors. If $AB = BA$, then there is a basis of $\mathbb{H}$ formed by common eigenvectors of A and B, useful in the study of the operators which can be expressed as a function of A and B. Such a basis does not exist if $AB \neq BA$, but a weaker version of this approach is possible by using a frame. By starting from an orthonormal basis $\{\varphi_i\}_{i=1}^{M}$ formed by eigenvectors of A and an orthonormal basis $\{\psi_j\}_{j=1}^{M}$ formed by eigenvectors of B we can restrict ourselves to a subspace of the form

$$\mathcal{H} = \text{span}\{\varphi_1, \varphi_2, \ldots, \varphi_N\},$$

and use the frame $\{\pi\psi_j\}_{j=1}^M$, where π is the orthogonal projector corresponding to $\mathcal{H}$. In order to illustrate this method, let $\mathbb{Z}_M = \mathbb{Z}/M\mathbb{Z} = \{0, 1, \ldots, M-1\}$ and $A, B : l^2(\mathbb{Z}_M) \longrightarrow l^2(\mathbb{Z}_M)$ be the linear operators defined in terms of the canonical basis $\{\delta_i\}_{i=1}^M$ as

$$A\delta_j = \delta_{j-1}, \qquad B\delta_j = \mathrm{e}^{\frac{2\pi \mathrm{i}}{M}j}\delta_j$$

(the elements of $\mathbb{Z}_M$ are integers considered modulo M, and particularly, $-1 = M - 1$).

The elements of the canonical basis $\{\delta_i\}_{i=1}^M$ are eigenfunctions of B. The functions $\phi_0, \phi_1, \ldots, \phi_{M-1} : \mathbb{Z}_M \longrightarrow \mathbb{C}$ defined as

$$\phi_j(k) = \frac{1}{\sqrt{M}}\,\mathrm{e}^{-\frac{2\pi \mathrm{i}}{M}jk}, \tag{55}$$

that is,

$$\phi_j = \frac{1}{\sqrt{M}}\sum_{k=0}^{M-1}\mathrm{e}^{-\frac{2\pi \mathrm{i}}{M}jk}\delta_k$$

are eigenfunctions of A

$$A\phi_j = \frac{1}{\sqrt{M}}\sum_{k=0}^{M-1}\mathrm{e}^{-\frac{2\pi \mathrm{i}}{M}jk}\delta_{k-1} = \mathrm{e}^{-\frac{2\pi \mathrm{i}}{M}j}\phi_j,$$

and form an orthonormal basis in $l^2(\mathbb{Z}_M)$. Let $N \leqslant M$ and $\mathcal{H} = \mathrm{span}\{\phi_0, \phi_1, \ldots, \phi_{N-1}\}$. The elements

$$|u_j\rangle = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\mathrm{e}^{\frac{2\pi \mathrm{i}}{M}jk}\phi_k \qquad j \in \{0, 1, \ldots, M-1\} \tag{56}$$

form a frame in the subspace $\mathcal{H}$ such that

$$\frac{N}{M}\sum_{j=0}^{M-1}|u_j\rangle\langle u_j| = \mathbb{I}_{\mathcal{H}}.$$

If $j \neq k$, then

$$\langle u_j|u_k\rangle = \frac{1}{N}\sum_{p=0}^{N-1}\mathrm{e}^{\frac{2\pi \mathrm{i}}{M}(k-j)p} = \frac{1}{N}\frac{1-\mathrm{e}^{\frac{2\pi \mathrm{i}}{M}(k-j)N}}{1-\mathrm{e}^{\frac{2\pi \mathrm{i}}{M}(k-j)}} = \frac{\mathrm{e}^{\frac{\pi \mathrm{i}}{M}(k-j)(n-1)}}{N}\frac{\sin\frac{n\pi}{M}(k-j)}{\sin\frac{\pi}{M}(k-j)}.$$

According to the quantization scheme defined in subsection 4.1, the considered frame allows us to associate with each function $f : \mathbb{Z}_M \longrightarrow \mathbb{R}$ the operator

$$A_f : \mathcal{H} \longrightarrow \mathcal{H}, \qquad A_f = \frac{N}{M}\sum_{k=0}^{M-1} f(k)|u_k\rangle\langle u_k|$$

having the lower symbol

$$\begin{aligned}\check{f}(j) = \langle u_j|A_f|u_j\rangle &= \frac{N}{M}\sum_{k=0}^{M-1} f(k)|\langle u_j|u_k\rangle|^2 \\ &= \frac{N}{M}f(j) + \frac{N}{M}\sum_{k\neq j} f(k)|\langle u_j|u_k\rangle|^2 \\ &= \frac{N}{M}f(j) + \frac{1}{NM}\sum_{k\neq j} f(k)\frac{\sin^2\frac{n\pi}{M}(k-j)}{\sin^2\frac{\pi}{M}(k-j)}.\end{aligned}$$

The entries of the matrix of A_f in the orthonormal basis $\{|\phi_0\rangle, |\phi_1\rangle, \ldots, |\phi_{N-1}\rangle\}$ are

$$\langle\phi_p|A_f|\phi_q\rangle = \frac{1}{M}\sum_{k=0}^{M-1} \mathrm{e}^{\frac{2\pi\mathrm{i}}{M}k(p-q)} f(k), \tag{57}$$

and

$$\begin{aligned}\langle u_j|A_f|u_j\rangle &= \sum_{p,q=0}^{N-1} \langle u_j|\phi_p\rangle\langle\phi_p|A_f|\phi_q\rangle\langle\phi_q|u_j\rangle \\ &= \frac{1}{N}\sum_{p,q=0}^{N-1} \mathrm{e}^{\frac{2\pi\mathrm{i}}{M}(q-p)j}\langle\phi_p|A_f|\phi_q\rangle.\end{aligned} \tag{58}$$

Particularly, we have

$$\langle\phi_p|A_f|\phi_q\rangle = \frac{1}{M}\frac{1-a^M}{1-a\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{M}(p-q)}} \qquad \text{in the case} \quad f(k) = a^k$$

and

$$\langle\phi_p|A_f|\phi_q\rangle = \frac{1}{M}(1+\mathrm{e}^{\frac{2\pi\mathrm{i}}{M}(p-q)})^{M-1} \quad \text{in the case} \quad f(k) = \binom{M-1}{k}.$$

It is known that the functions $f_j : \mathbb{Z}_m \longrightarrow \mathbb{C}$ defined in terms of Hermite polynomials

$$f_j(k) = \sum_{l=-\infty}^{\infty} \mathrm{e}^{-\frac{\pi}{M}(lM+k)^2} H_j\left(\sqrt{\frac{2\pi}{M}}(lM+k)\right)$$

are eigenfunctions of the discrete Fourier transform [36]

$$\frac{1}{\sqrt{M}}\sum_{p=0}^{M-1} \mathrm{e}^{\frac{2\pi\mathrm{i}}{M}pk} f_j(p) = \mathrm{i}^j f_j(k).$$

Therefore,

$$\langle\phi_p|A_{f_j}|\phi_q\rangle = \frac{\mathrm{i}^j}{\sqrt{M}} f_j(p-q).$$

If the real number x is not a multiple of M, then

$$\sum_{k=0}^{M-1} \mathrm{e}^{\frac{2\pi\mathrm{i}}{M}kx} = \frac{1-\mathrm{e}^{2\pi\mathrm{i}x}}{1-\mathrm{e}^{\frac{2\pi\mathrm{i}}{M}x}}.$$

By differentiating this relation we get

$$\sum_{k=0}^{M-1} k\,\mathrm{e}^{\frac{2\pi\mathrm{i}}{M}kx} = \frac{-M\,\mathrm{e}^{2\pi\mathrm{i}x}\left(1-\mathrm{e}^{\frac{2\pi\mathrm{i}}{M}x}\right)+\mathrm{e}^{\frac{2\pi\mathrm{i}}{M}x}(1-\mathrm{e}^{2\pi\mathrm{i}x})}{\left(1-\mathrm{e}^{\frac{2\pi\mathrm{i}}{M}x}\right)^2}$$

whence,

$$\langle\phi_p|A_f|\phi_q\rangle = \begin{cases} \frac{M-1}{2} & \text{if} \quad p=q \\ \frac{1}{\mathrm{e}^{\frac{2\pi\mathrm{i}}{M}(p-q)}-1} & \text{if} \quad p\neq q \end{cases} \qquad \text{in the case } f(k) = k.$$

4.5. Finite quantum systems

The study of quantum systems described by finite-dimensional spaces was initiated by Weyl [45] and Schwinger [42] and rely upon the discrete Fourier transform. Let n be a fixed positive integer. The set $\mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}_n$ considered together with the multiplication law

$$(\theta, \alpha, \beta)(\theta', \alpha', \beta') = (\theta + \theta' + \beta\alpha', \alpha + \alpha', \beta + \beta')$$

where all sums are modulo n, is a group. This group of order n^3 is regarded as a discrete version of the Heisenberg group [44].

In any n-dimensional Hilbert space $\mathcal{H}$, we can define by choosing an orthonormal basis $\{|0\rangle, |1\rangle, \ldots, |n-1\rangle\}$ the Weyl operators $A, B : \mathcal{H} \longrightarrow \mathcal{H}$

$$A|j\rangle = |j-1\rangle, \qquad B|j\rangle = \mathrm{e}^{\frac{2\pi\mathrm{i}}{n}j}|j\rangle$$

satisfying the relation

$$A^\alpha B^\beta = \mathrm{e}^{\frac{2\pi\mathrm{i}}{n}\alpha\beta} B^\beta A^\alpha.$$

The mapping

$$(\theta, \alpha, \beta) \mapsto \mathrm{e}^{\frac{2\pi\mathrm{i}}{n}\theta} A^\alpha B^\beta$$

defines a unitary irreducible representation of the discrete Heisenberg group in $\mathcal{H}$ and for any vector $|v\rangle = \sum_{k=0}^{n-1} v_k|k\rangle$ we have

$$\mathrm{e}^{\frac{2\pi\mathrm{i}}{n}\theta} A^\alpha B^\beta |v\rangle = \mathrm{e}^{\frac{2\pi\mathrm{i}}{n}(\theta+\alpha\beta)} \sum_{k=0}^{n-1} \mathrm{e}^{\frac{2\pi\mathrm{i}}{n}\beta k} v_{k+\alpha}|k\rangle.$$

By choosing a unit vector $|u\rangle = \sum_{k=0}^{n-1}\mu_k|k\rangle$ with stationary group $G_u = \mathbb{Z}_n \times \{0\} \times \{0\}$ and neglecting the phase factors we get the frame [46]

$$\left\{ |\alpha, \beta\rangle = \sum_{k=0}^{n-1} \mathrm{e}^{\frac{2\pi\mathrm{i}}{n}\beta k}\mu_{k+\alpha}|k\rangle \,\middle|\, (\alpha, \beta) \in \mathbb{Z}_n \times \mathbb{Z}_n \right\}, \tag{59}$$

and the resolution of identity

$$\frac{1}{n}\sum_{\alpha,\beta=0}^{n-1} |\alpha, \beta\rangle\langle\alpha, \beta| = \mathbb{I}_{\mathcal{H}}. \tag{60}$$

In the case $n = 3$, by choosing $|u\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ we obtain the frame

$$\begin{array}{lll}
|0,0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle, & |0,1\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}\varepsilon|1\rangle, & |0,2\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}\varepsilon^2|1\rangle \\
|1,0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|2\rangle, & |1,1\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}\varepsilon^2|2\rangle, & |1,2\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}\varepsilon|2\rangle \\
|2,0\rangle = \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|2\rangle, & |2,1\rangle = \frac{1}{\sqrt{2}}\varepsilon|1\rangle + \frac{1}{\sqrt{2}}\varepsilon^2|2\rangle, & |2,2\rangle = \frac{1}{\sqrt{2}}\varepsilon^2|1\rangle + \frac{1}{\sqrt{2}}\varepsilon|2\rangle
\end{array}$$

where $\varepsilon = \mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}$.

The set $\mathbb{Z}_n \times \mathbb{Z}_n$ can be regarded as a finite version of the phase space, and with each *classical observable* $f : \mathbb{Z}_n \times \mathbb{Z}_n \longrightarrow \mathbb{R}$ we associate the linear operator

$$A_f : \mathcal{H} \longrightarrow \mathcal{H}, \qquad A_f = \frac{1}{n}\sum_{\alpha,\beta=0}^{n-1} f(\alpha, \beta)|\alpha, \beta\rangle\langle\alpha, \beta|. \tag{61}$$

For example, in the case $n = 2$ by starting from $|u\rangle = \frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle$ we get the frame

$$|0,0\rangle = \tfrac{3}{5}|0\rangle + \tfrac{4}{5}|1\rangle, \qquad |0,1\rangle = \tfrac{3}{5}|0\rangle - \tfrac{4}{5}|1\rangle$$
$$|1,0\rangle = \tfrac{4}{5}|0\rangle + \tfrac{3}{5}|1\rangle, \qquad |1,1\rangle = \tfrac{4}{5}|0\rangle - \tfrac{3}{5}|1\rangle$$

and with each function $f : \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow \mathbb{R}$ we associate the operator

$$\begin{aligned} A_f &= \frac{1}{2}\sum_{\alpha,\beta=0}^{1} f(\alpha,\beta)|\alpha,\beta\rangle\langle\alpha,\beta| \\ &= \frac{1}{50}\left\{ f(0,0)\begin{pmatrix} 9 & 12 \\ 12 & 16 \end{pmatrix} + f(1,0)\begin{pmatrix} 16 & 12 \\ 12 & 9 \end{pmatrix} \right. \\ &\quad \left. + f(0,1)\begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix} + f(1,1)\begin{pmatrix} 16 & -12 \\ -12 & 9 \end{pmatrix} \right\}. \end{aligned}$$

We have $\langle 0,0|0,0\rangle = \langle 1,0|1,0\rangle = \langle 0,1|0,1\rangle = \langle 1,1|1,1\rangle = 1$ and

$$\begin{aligned} &\langle 0,0|0,1\rangle = -\tfrac{7}{25} && \langle 0,1|1,0\rangle = 0 \\ &\langle 0,0|1,0\rangle = \tfrac{24}{25} && \langle 0,1|1,1\rangle = \tfrac{24}{25} \\ &\langle 0,0|1,1\rangle = 0 && \langle 1,0|1,1\rangle = \tfrac{7}{25} \end{aligned}$$

and the lower symbol is

$$\begin{aligned} \langle 0,0|A_f|0,0\rangle &= \tfrac{1}{2}\left\{ f(0,0) + f(0,1)\left(\tfrac{7}{25}\right)^2 + f(1,0)\left(\tfrac{24}{25}\right)^2 \right\} \\ \langle 0,1|A_f|0,1\rangle &= \tfrac{1}{2}\left\{ f(0,0)\left(\tfrac{7}{25}\right)^2 + f(0,1) + f(1,1)\left(\tfrac{24}{25}\right)^2 \right\} \\ \langle 1,0|A_f|1,0\rangle &= \tfrac{1}{2}\left\{ f(0,0)\left(\tfrac{24}{25}\right)^2 + f(1,0) + f(1,1)\left(\tfrac{7}{25}\right)^2 \right\} \\ \langle 1,1|A_f|1,1\rangle &= \tfrac{1}{2}\left\{ f(0,1)\left(\tfrac{24}{25}\right)^2 + f(1,0)\left(\tfrac{7}{25}\right)^2 + f(1,1) \right\}. \end{aligned}$$

One can remark that the lower symbols corresponding to the classical observables we have to analyze depend on the fiducial vector. Therefore, the fiducial vector we use must be a privileged one, for example, a kind of fundamental state. We should also notice the way the values of the observables are 'redistributed' along the probability distribution.

4.6. An application of the frame quantization to crystals

The set $\mathbb{Z} \times \mathbb{Z}$ can be regarded as a mathematical model for a two-dimensional crystal. By imposing the cyclic boundary condition, the space $\mathcal{E} = l^2(\mathbb{Z}_N \times \mathbb{Z}_N)$ and the operator

$$\begin{aligned} &H : \mathcal{E} \longrightarrow \mathcal{E}, \\ &(H\psi)(n_1,n_2) = \psi(n_1+1,n_2) + \psi(n_1-1,n_2) + \psi(n_1,n_2+1) + \psi(n_1,n_2-1) \end{aligned} \tag{62}$$

allow one to describe the electron evolution inside the crystal in the tight binding approximation [38]. For any $k = (k_1,k_2) \in \mathbb{Z}_N \times \mathbb{Z}_N$, the function

$$\psi_k : \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow \mathbb{C}, \qquad \psi_k(n_1,n_2) = \mathrm{e}^{\frac{2\pi\mathrm{i}}{N}(k_1n_1+k_2n_2)} \tag{63}$$

is an eigenfunction of H corresponding to the eigenvalue

$$E_k = \mathrm{e}^{\frac{2\pi\mathrm{i}}{N}k_1} + \mathrm{e}^{-\frac{2\pi\mathrm{i}}{N}k_1} + \mathrm{e}^{\frac{2\pi\mathrm{i}}{N}k_2} + \mathrm{e}^{-\frac{2\pi\mathrm{i}}{N}k_2} = 2\cos\frac{2\pi}{N}k_1 + 2\cos\frac{2\pi}{N}k_2, \tag{64}$$

that is,

$$H\psi_k = E_k\psi_k.$$

One can remark that

$$E_k = \sum_{(n_1,n_2)\in\mathcal{C}} \psi_k(n_1, n_2),$$

where $\mathcal{C}$ is the cluster

$$\mathcal{C} = \{(1, 0), (-1, 0), (0, 1), (0, -1)\} \subset \mathbb{Z}_N \times \mathbb{Z}_N.$$

The Hilbert space $l^2(\mathcal{C})$ can be identified with the subspace

$$\mathcal{H} = \{\varphi : \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow \mathbb{C} | \varphi(n_1, n_2) = 0 \text{ for } (n_1, n_2) \notin \mathcal{C}\}.$$

The N^2 functions $\left\{\left|\delta_{(n_1,n_2)}\right\rangle = \delta_{(n_1,n_2)} : \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow \mathbb{C}\right\}_{n_1,n_2\in\mathbb{Z}_N}$

$$\delta_{(n_1,n_2)}(n_1', n_2') = \begin{cases} 1 & \text{if} \quad (n_1', n_2') = (n_1, n_2) \\ 0 & \text{if} \quad (n_1', n_2') \neq (n_1, n_2), \end{cases}$$

and the N^2 functions $\left\{\left|\psi_{(k_1,k_2)}\right\rangle = \psi_{(k_1,k_2)} : \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow \mathbb{C}\right\}_{k_1,k_2\in\mathbb{Z}_N}$

$$\psi_{(k_1,k_2)}(n_1, n_2) = \frac{1}{N} \, \mathrm{e}^{\frac{2\pi \mathrm{i}}{N}(k_1 n_1 + k_2 n_2)} \tag{65}$$

form two orthonormal bases of $\mathcal{E}$ related through the discrete Fourier transform.

The orthogonal projector corresponding to $\mathcal{H}$ is

$$\pi = \sum_{(n_1,n_2)\in\mathcal{C}} \left|\delta_{(n_1,n_2)}\right\rangle\!\left\langle\delta_{(n_1,n_2)}\right|,$$

and in view of theorem 1, the N^2 functions $\{|k_1, k_2\rangle : \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow \mathbb{C}\}_{k_1,k_2\in\mathbb{Z}_N}$

$$|k_1, k_2\rangle = \frac{1}{2} \sum_{(n_1,n_2)\in\mathcal{C}} \left|\delta_{(n_1,n_2)}\right\rangle\!\left\langle\delta_{(n_1,n_2)}\middle|\psi_{(k_1,k_2)}\right\rangle = \frac{1}{2} \sum_{(n_1,n_2)\in\mathcal{C}} \mathrm{e}^{\frac{2\pi \mathrm{i}}{N}(k_1 n_1 + k_2 n_2)} \left|\delta_{(n_1,n_2)}\right\rangle$$

form a frame in $\mathcal{H}$

$$\frac{4}{N^2} \sum_{k_1,k_2=0}^{N-1} |k_1, k_2\rangle\langle k_1, k_2| = \mathbb{I}_{\mathcal{H}}.$$

They satisfy the relation

$$\begin{aligned} \langle k_1, k_2 | k_1', k_2'\rangle &= \frac{1}{4} \sum_{(n_1,n_2)\in\mathcal{C}} \mathrm{e}^{\frac{2\pi \mathrm{i}}{N}[(k_1'-k_1)n_1 + (k_2'-k_2)n_2]} \\ &= \frac{1}{2}\left[\cos\frac{2\pi}{N}(k_1' - k_1) + \cos\frac{2\pi}{N}(k_2' - k_2)\right]. \end{aligned}$$

With a *classical* observable defined by $f : \mathbb{Z}_N \times \mathbb{Z}_N \longrightarrow \mathbb{R}$ we associate the linear operator

$$A_f : \mathcal{H} \longrightarrow \mathcal{H}, \qquad A_f = \frac{4}{N^2} \sum_{k_1,k_2=0}^{N-1} f(k_1, k_2) |k_1, k_2\rangle\langle k_1, k_2|, \tag{66}$$

with the lower symbol

$$\langle k_1, k_2 | A_f | k_1, k_2\rangle = \frac{1}{N^2} \sum_{k_1',k_2'=0}^{N-1} f(k_1', k_2') \left[\cos\frac{2\pi}{N}(k_1' - k_1) + \cos\frac{2\pi}{N}(k_2' - k_2)\right]^2.$$

In the case of frame quantization we analyze a classical observable by using a suitable smaller dimensional subspace. We can increase the resolution of our analysis by choosing a larger cluster including second-order or second- and third-order neighbors of (0, 0).

4.7. Quantization with finite tight frames overcomplete by one vector

For each positive integer n we consider in the Euclidean space $\mathbb{R}^{n+1}$ the hyperspace

$$\mathcal{H}_n = \{x = (x_0, x_1, \ldots, x_n) | x_0 + x_1 + \cdots + x_n = 0\}.$$

The orthogonal projector corresponding to $\mathcal{H}_n$ is $\pi : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$,

$$\pi(x_0, x_1, \ldots, x_n) = \left(\frac{n x_0 - x_1 - \cdots - x_n}{n+1}, \frac{-x_0 + n x_1 - x_2 - \cdots - x_n}{n+1}, \ldots, \frac{-x_0 - \cdots - x_{n-1} + n x_n}{n+1} \right)$$

and the orthogonal projections of the vectors of the canonical basis

$$w_0 = \pi(1, 0, 0, \ldots, 0) = \left(\frac{n}{n+1}, -\frac{1}{n+1}, -\frac{1}{n+1}, \ldots, -\frac{1}{n+1} \right)$$
$$w_1 = \pi(0, 1, 0, \ldots, 0) = \left(-\frac{1}{n+1}, \frac{n}{n+1}, -\frac{1}{n+1}, \ldots, -\frac{1}{n+1} \right)$$
$$\vdots$$
$$w_n = \pi(0, 0, \ldots, 0, 1) = \left(-\frac{1}{n+1}, -\frac{1}{n+1}, \ldots, -\frac{1}{n+1}, \frac{n}{n+1} \right)$$

have the same norm

$$\|w_0\| = \|w_1\| = \cdots = \|w_n\| = \sqrt{\frac{n}{n+1}}.$$

The corresponding normalized vectors

$$|u_0\rangle = \frac{w_0}{\|w_0\|} = \left(\sqrt{\frac{n}{n+1}}, -\frac{1}{\sqrt{n(n+1)}}, -\frac{1}{\sqrt{n(n+1)}}, \ldots, -\frac{1}{\sqrt{n(n+1)}} \right)$$
$$|u_1\rangle = \frac{w_1}{\|w_1\|} = \left(-\frac{1}{\sqrt{n(n+1)}}, \sqrt{\frac{n}{n+1}}, -\frac{1}{\sqrt{n(n+1)}}, \ldots, -\frac{1}{\sqrt{n(n+1)}} \right)$$
$$\vdots$$
$$|u_n\rangle = \frac{w_n}{\|w_n\|} = \left(-\frac{1}{\sqrt{n(n+1)}}, -\frac{1}{\sqrt{n(n+1)}}, \ldots, -\frac{1}{\sqrt{n(n+1)}}, \sqrt{\frac{n}{n+1}} \right)$$

form a normalized tight frame

$$\frac{n}{n+1} \sum_{k=0}^{n} |u_k\rangle\langle u_k| = \mathbb{I}_{\mathcal{H}_n},$$

such that

$$\langle u_k | u_j \rangle = -\frac{1}{n} \qquad \text{for} \quad k \neq j.$$

With each function $f : \{0, 1, \ldots, n\} \longrightarrow \mathbb{R}$ we associate the linear operator

$$A_f : \mathcal{H}_n \longrightarrow \mathcal{H}_n, \qquad A_f = \frac{n}{n+1} \sum_{k=0}^{n} f(k) |u_k\rangle\langle u_k|.$$

The corresponding lower symbol is the function $\check{f}_n : \{0, 1, \ldots, n\} \longrightarrow \mathbb{R}$,

$$\check{f}_n(j) = \langle u_j | A_f | u_j \rangle = \frac{n-1}{n} f(j) + \frac{1}{n(n+1)} \sum_{k=0}^{n} f(k)$$

and if $f : \{0, 1, 2, \dots\} \longrightarrow \mathbb{R}$ is a bounded function then we have

$$\lim_{n\to\infty} \check{f}_n(j) = f(j), \qquad \text{for any} \quad j \in \{0, 1, 2, \dots\},$$

as expected from the general results given in subsection 4.3. More than that, if $f, g : \{0, 1, 2, \dots\} \longrightarrow \mathbb{R}$ are two bounded functions then

$$\begin{aligned}\langle u_j|[A_f, A_g]|u_j\rangle &= \frac{n^2}{(n+1)^2}\sum_{k=0}^{n}\sum_{l=0}^{n}(f(k)g(l) - f(l)g(k))\langle u_j|u_k\rangle\langle u_k|u_l\rangle\langle u_l|u_j\rangle\\ &= -\frac{1}{n(n+1)^2}\sum_{k\neq j}\sum_{l\neq j}(f(k)g(l) - f(l)g(k))\end{aligned}$$

and the lower symbol of the commutator $[A_f, A_g]$ has the property

$$\lim_{n\to\infty} \langle u_j|[A_f, A_g]|u_j\rangle = 0, \qquad \text{for any} \quad j \in \{0, 1, 2, \dots\}.$$

5. Conclusions

In this review we have presented some elements concerning certain applications of finite frames to crystal/quasicrystal physics and to quantum physics. In order to achieve these two main objectives and inspired by the analogy with standard coherent states, we have introduced the notion of normalized Parseval frame, directly related to the notion of Parseval frame, and analyzed some stochastic aspects. In particular, we have defined two types of 'distances', $\eta = r - 1$ and $\zeta = 1 - \min_{1\leqslant i\leqslant M} \kappa_i$, between frames and orthonormal basis in the superspace. For the applications to crystals and quasicrystals, based on the embedding into a superspace defined by a frame, we have analyzed the subset of the elements which can be represented as a linear combination of frame vectors by using only integer coefficients. In this way we have identified two important classes of tight frames, namely the periodic frames and the quasiperiodic frames. We have also presented some convergent sequences of finite frames and an example of continuous deformation of a periodic tetrahedral frame into an icosahedral quasiperiodic frame. The stochastic aspects of the finite frame quantization and most of the considerations concerning the application of finite frames to crystals/quasicrystals seem to be new, and might be regarded as a contribution to the finite frame theory.

The description of the elements of a vector space based on the use of an overcomplete system is a general method rediscovered several times in different areas of mathematics, science and engineering. For example, in crystallography there exists an alternative description for the hexagonal crystals based on the use of an additional axis. We show that the use of a frame leads to a simpler description of atomic positions in a diamond type crystal. This leads to a simpler description of the symmetry transformations and of the mathematical objects with physical meaning. Some of the most important models used in quasicrystal physics can be generated in a unitary way by using the imbedding into a superpace defined by certain frames. These observations allow a fructuous interchange of ideas and methods between frame theory and quasicrystal physics.

Finite frame quantization replaces a real function f defined on a finite set by a self-adjoint operator A_f, and the eigenvalues of A_f can be regarded as the 'quantum spectrum' of f. We compare f with the mean values of A_f corresponding to the frame vectors, in the general case and in several particular cases. We have explained the role of the parameter ζ as a kind of distance of the quantum non-commutative world to the classical commutative one. The notion of normalized Parseval frame and the corresponding quantization of discrete variable functions contains many questions which deserve to be thoroughly investigated in order that they might shed light on a better understanding of quantum mechanics and quantization.

Acknowledgments

NC acknowledges the support provided by CNCSIS under the grant IDEI 992-31/2007 and thanks laboratoire APC, Université Paris Diderot Paris 7, for their kind hospitality during the time this work was carried out.

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